

Large Deviation for Reflected Backward Stochastic Differential Equations

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Abstract

In this note, we prove the Freidlin-Wentzell's large deviation principle for BSDEs with one-sided reflection.

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1 Introduction

Backward stochastic differential equations (BSDEs in short) with reflection were firstly studied by El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [3], which is, a standard BSDEs with an additional continuous, increasing process to keep the solution above a certain given continuous boundary process. This increasing process must be chosen in certain minimal way, i.e. an integral condition, called Skorohod reflecting condition, is satisfied. Besides, they gave a probabilistic interpretation of viscosity solution of variation inequality by the solution of reflected BSDEs. As we have known, it can be used widely in mathematical finance, for example, American options in incomplete market (see [6]).

On the other hand, the large deviation principle (LDP) characterizes the limiting behavior of probability measure in term of rate function which is a very active field in applied probability and largely used in rare events simulation. Recently, there has been a growing literature on studying the applications of LDP in finance (see [7]).

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We now consider the following small perturbation of reflected forward and backward stochastic differential equations (1.1) and (1.2)

$$X^{\varepsilon,t,x}(s) = x + \int_t^s b(r, X^{\varepsilon,t,x}(r)) dr + \varepsilon^{\frac{1}{2}} \int_t^s dW(r) \quad (1.1)$$

$$\begin{cases} Y^{\varepsilon,t,x}(s) = g(X^{\varepsilon,t,x}(T)) + \int_s^T f(r, X^{\varepsilon,t,x}(r), Y^{\varepsilon,t,x}(r), Z^{\varepsilon,t,x}(r)) dr \\ \quad + K^{\varepsilon,t,x}(T) - K^{\varepsilon,t,x}(r) - \int_s^T Z^{\varepsilon,t,x}(r) dW(r) \\ Y^{\varepsilon,t,x}(s) \geq h(s, X^{\varepsilon,t,x}(s)), \quad t \leq s \leq T \\ \int_t^T (Y^{\varepsilon,t,x}(s) - h(s, X^{\varepsilon,t,x}(s))) dK^{\varepsilon,t,x}(s), \end{cases} \quad (1.2)$$

The solution of this equation is denoted by

$$(X^{\varepsilon,t,x}(s), Y^{\varepsilon,t,x}(s), Z^{\varepsilon,t,x}(s), K^{\varepsilon,t,x}(s), t \leq s \leq T).$$

We want to establish the large deviation principle of the law of $Y^{\varepsilon,t,x}$ in the space of $C([0, T]; \mathbf{R}^n)$, namely the asymptotic estimates of probabilities $P(Y^{\varepsilon,t,x} \in \Gamma)$, where $\Gamma \in \mathcal{B}(C([0, T]; \mathbf{R}^n))$.

In [8], Rainero first considered the same small random perturbation for BSDEs and obtained the Freidlin-Wentzell's large deviation estimates in $C([0, T]; \mathbf{R}^n)$ using the contraction principle. Subsequently, Essaky in [4] investigated the large deviation for BSDEs with subdifferential operator. It is necessary to point out that BSDEs with subdifferential operator include as a special case BSDEs whose solution is reflected at the boundary of a convex subset of \mathbf{R}^n (for more information see [5]). Moreover, their convex function is fixed. Besides, in [4] b does not depends on time variable. From this viewpoint, our work cannot be covered by their results.

In Section 2, we give the framework of our paper. Here we review the basic concepts of large deviation and assumptions on (1.1) and (1.2). Then in Section 3 we show our main result Theorem 8. Throughout the paper, C and K , with or without indexes will denote different constants changing from line to line whose values are not important.

2 Preliminaries

Let us begin by introducing the setting for the stochastic differential differential we want to investigate. Consider as Brownian motion W is the n -dimensional coordinate process on the classical Wiener space (Ω, \mathcal{F}, P) , i.e., Ω is the set of continuous functions from $[0, T]$ to \mathbf{R}^n starting from 0 ($\Omega = C([0, T]; \mathbf{R}^n)$), \mathcal{F} the completed Borel σ -algebra over Ω , P the Wiener measure and W the canonical process: $W_s(\omega) = \omega_s$, $s \in [0, T]$, $\omega \in \Omega$. By $\{\mathcal{F}_s, 0 \leq s < T\}$ we denote the natural filtration generated by $\{W_s\}_{0 \leq s < T}$ and augmented by all P -null sets, i.e.,

$$\mathcal{F}_s = \sigma\{W_r, r \leq s\} \vee \mathcal{N}_p, \quad s \in [0, T],$$

where \mathcal{N}_p is the set of all P -null subsets. For any $n \geq 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbf{R}^n$.

b, g, f and h in Eq. (1.1) and (1.2) are defined as follows. First, let $b : [0, T] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ be continuous mapping and satisfy linear growth, which are Lipschitz with respect to their second variable, uniformly with respect to $t \in [0, T]$. Second, $g \in C(\mathbf{R}^n)$ and has at most polynomial growth at infinity and satisfies Lipschitz condition. For f , assume that $f : [0, T] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$ is jointly continuous and for some $K > 0$, admits that

$$|f(t, x, 0, 0)| \leq K(1 + |x|^2), \quad (2.1)$$

$$\left| f(t, x, y, z) - f(t, x', y', z') \right| \leq K \left(|x - x'| + |y - y'| + |z - z'| \right), \quad (2.2)$$

for $t \in [0, T]$, $x, x', z, z' \in \mathbf{R}^n$, $y, y' \in \mathbf{R}$. Finally, $h : [0, T] \times \mathbf{R}^n \rightarrow \mathbf{R}$ is jointly continuous in t and x and satisfies

$$h(t, x) \leq K(1 + |x|), \quad t \in [0, T], \quad x \in \mathbf{R}^n, \quad (2.3)$$

$$\left| h(t, x) - h(t, x') \right| \leq K|x - x'|, \quad x, x' \in \mathbf{R}^n. \quad (2.4)$$

We assume moreover that $g(x) \geq h(T, x)$, $x \in \mathbf{R}^n$

For each $t > 0$, we denote by $\{\mathcal{F}_s^t, t \leq s \leq T\}$ the natural filtration of the Brownian motion $\{B_s - B_t, t \leq s \leq T\}$, augmented by the P -null set of \mathcal{F} . It follows from Theorem 5.2 in [3] that, under the above assumptions, there exists a unique triple $(Y^{\varepsilon, t, x}, Z^{\varepsilon, t, x}, K^{\varepsilon, t, x})$ of $\{\mathcal{F}_s^t\}$ progressively measurable processes, which solves Eq. (1.1) and (1.2).

Now we give two more accurate estimates on the norm of the solution similar to Proposition 3.5 in [3].

Lemma 1. *Let $(X^{\varepsilon, t, x}(s), Y^{\varepsilon, t, x}(s), Z^{\varepsilon, t, x}(s), K^{\varepsilon, t, x}(s))$ $t \leq s \leq T$, $\varepsilon > 0$) be the solution of the above reflected FBSDE (1.1), (2.2), then there exists a constant C such that*

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \leq s \leq T} |Y^{\varepsilon, t, x}(s)|^2 + \int_0^T |Z^{\varepsilon, t, x}(s)|^2 + (K^{\varepsilon, t, x})^2(T) \right] \\ & \leq C \mathbb{E} \left[g^2(X^{\varepsilon, t, x}(T)) + \int_t^T f^2(s, X^{\varepsilon, t, x}(s), 0, 0) dt + \sup_{t \leq s \leq T} h^2(t, X^{\varepsilon, t, x}(s)) \right] \end{aligned} \quad (2.5)$$

We now consider the following related obstacle problem for a parabolic PDEs. More precisely, a solution of the obstacle problem is a function $u : [0, T] \times \mathbf{R}^n \rightarrow \mathbf{R}$ which satisfies

$$\begin{cases} \min \left(u(t, x) - h(t, x), -\frac{\partial u}{\partial t}(t, x) - \mathcal{L}_t u(t, x) - f\left(t, x, u(t, x), \left(\nabla u \varepsilon^{\frac{1}{2}}\right)(t, x)\right) \right) = 0, \\ u(T, x) = g(x), \quad (t, x) \in (0, T) \times \mathbf{R}^n, \quad x \in \mathbf{R}^n, \end{cases} \quad (2.6)$$

where

$$\mathcal{L}_t = \frac{\varepsilon}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t, x) \frac{\partial}{\partial x_i}.$$

Now define

$$u^\varepsilon(t, x) = Y^{\varepsilon, t, x}(t), \quad (t, x) \in [0, T] \times \mathbf{R}^n, \quad \varepsilon > 0, \quad (2.7)$$

which is a deterministic quantity since it is \mathcal{F}_t^t measurable.

Definition 2. Let $t \in [0, T]$, we define the mapping $G^\varepsilon : C([t, T] : \mathbf{R}^n) \rightarrow C([t, T] : \mathbf{R}^n)$ by

$$G^\varepsilon(\psi) = [t \rightarrow u^\varepsilon(t, \psi(t))], \quad 0 \leq t \leq s \leq T, \quad \psi \in C([t, T] : \mathbf{R}^n),$$

where u^ε is given by (2.7).

Immediately, we have

$$Y^{\varepsilon, t, x}(t) = G^\varepsilon(X^{\varepsilon, t, x})(t).$$

From now on, for $\varepsilon = 0$, u and G stand for u^0 and G^0 .

In order to prove the uniform convergence of the mapping G^ε , which will be shown in next section, we need to estimate the following formula

$$\|G^\varepsilon(\varphi) - G(\varphi)\| = \sup_{0 \leq s \leq T} |u^\varepsilon(s, \varphi(s)) - u(s, \varphi(s))|, \quad \varphi \in C([0, T] : \mathbf{R}^n)$$

or

$$\|G^\varepsilon(\varphi) - G(\varphi)\| = \sup_{0 \leq s \leq T} |Y^{\varepsilon, s, \varphi(s)}(s) - Y^{s, \varphi(s)}(s)|$$

Proposition 3. Under the above assumptions (2.1)-(2.4), u^ε is a viscosity solution of the obstacle problem (2.6).

The proof can be seen in [3]. Before giving a large deviation principle for SDEs (1.1), we recall the following definitions.

Definition 4. If E is a complete separable metric space, then a function \mathcal{I} defined on E is called a rate function if it has the following properties:

$$\begin{cases} (a) & \mathcal{I} : E \rightarrow [0, +\infty], \mathcal{I} \text{ is lower semicontinuous.} \\ (b) & \text{If } 0 \leq a \leq \infty, \text{ then } C_I(a) = \{x \in E : \mathcal{I}(x) \leq a\} \text{ is compact.} \end{cases} \quad (2.8)$$

Definition 5. If E is a complete separable metric space, \mathcal{B} is the Borel σ -field on E , $\{\mu_\varepsilon : \varepsilon > 0\}$ is a family of probability measure on (E, \mathcal{B}) , and \mathcal{I} is a function defined on E and satisfying (2.8), then we say that $\{\mu_\varepsilon\}_{\varepsilon > 0}$ satisfies a large deviation principle with rate \mathcal{I} if:

$$\begin{cases} (a) & \text{For every open subset } A \text{ of } E, \\ & \liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mu_\varepsilon(A) \geq -\mathcal{I}(A). \\ (b) & \text{For every closed subset } A \text{ of } E, \\ & \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mu_\varepsilon(A) \leq -\mathcal{I}(A), \end{cases} \quad (2.9)$$

where, and below, if \mathcal{I} is a function defined on the set E and A is a subset of E , then $\mathcal{I}(A)$ is defined to be the infimum of \mathcal{I} on A . Unless otherwise stated, all lim infs and sups are as $\varepsilon \rightarrow 0$.

In the work of Michelle Boue and Paul Dupuis [1], they proved the large deviation principle for the solution of the SDEs (1.1) as follows:

Lemma 6. *The process $X^{\varepsilon,t,x}$ given by (1.1) satisfies a large deviation principle in $C([0, T] : \mathbf{R}^n)$ with rate function \mathcal{I} defined by*

$$\mathcal{I}_{t,x}(\xi) = \inf_{\{v \in L^2([0,T]:\mathbf{R}^n): \xi(t)=x+\int_t^s b(r,\xi(r))dr+\int_t^s v(r)dr\}} \frac{1}{2} \int_t^T \|v(s)\|^2 ds,$$

whenever $\{v \in L^2([0, T] : \mathbf{R}^n) : \xi(t) = x + \int_t^s b(r, \xi(r)) dr + \int_t^s v(r) dr\} \neq \emptyset$, and $\mathcal{I}_{t,x}(\xi) = \infty$ otherwise.

Remark 7. *Note that Laplace principle is equivalent to a large deviation principle if the definition of a rate function includes the requirement of compact level sets.*

3 Main Result

We have the following:

Theorem 8. *Under the assumptions (2.1)-(2.4), $Y^{\varepsilon,t,x}$ satisfies a large deviation principle with a rate function*

$$\tilde{\mathcal{I}}_{t,x}(\tilde{\xi}) = \inf \left\{ \mathcal{I}_{t,x}(\xi) \mid \tilde{\xi}(t) = G(\xi)(t) = u(t, \xi(t)), t \in [0, T], \xi \in C([0, T] : \mathbf{R}^n) \right\}. \quad (3.1)$$

For the proof of this theorem we need four auxiliary lemmata. Let $\chi^{t,x}$ be the solution of the following deterministic equation

$$\chi^{t,x}(s) = x + \int_t^s b(r, \chi^{t,x}(r)) dr. \quad (3.2)$$

We have the following

Lemma 9. *For all $\varepsilon \in (0, 1]$, there exists a constant $C > 0$, independent of x, t and ε , such that*

$$\mathbb{E} \left[\sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)| \right] \leq C\varepsilon. \quad (3.3)$$

Proof. Applying Itô's formula $(X^{\varepsilon,t,x}(s) - \chi^{t,x}(s))^2$ on $[t, T]$, we have

$$\begin{aligned} (X^{\varepsilon,t,x}(s) - \chi^{t,x}(s))^2 &= 2 \int_t^s (X^{\varepsilon,t,x}(r) - \chi^{t,x}(r)) [b(r, X^{\varepsilon,t,x}(r)) - b(r, \chi^{t,x}(r))] dr \\ &\quad + 2\sqrt{\varepsilon} \int_t^s (X^{\varepsilon,t,x}(r) - \chi^{t,x}(r)) dW(r) + \varepsilon \int_t^s dr \\ &\leq 2K \int_t^s |X^{\varepsilon,t,x}(r) - \chi^{t,x}(r)|^2 dr + \varepsilon T \\ &\quad + 2\sqrt{\varepsilon} \int_t^s (X^{\varepsilon,t,x}(r) - \chi^{t,x}(r)) dW(r), \end{aligned}$$

(3.4)

since b satisfies Lipchitz condition with Lipchitz constant K . Taking expectation on both sides of (3.4), we have

$$\mathbb{E} \left[|X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)|^2 \right] \leq 2K \int_t^s |X^{\varepsilon,t,x}(r) - \chi^{t,x}(r)|^2 dr + \varepsilon T.$$

It follows from Gronwall inequality that

$$\mathbb{E} \left[|X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)|^2 \right] \leq \varepsilon K T \exp(2K(s-t))$$

Hence

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)|^2 \right] &\leq 2K \int_t^T \mathbb{E} \left[|X^{\varepsilon,t,x}(r) - \chi^{t,x}(r)|^2 \right] dr + \varepsilon T \\ &\quad \mathbb{E} \left[\sup_{t \leq s \leq T} 2\sqrt{\varepsilon} \int_t^s (X^{\varepsilon,t,x}(r) - \chi^{t,x}(r)) dW(r) \right] \\ &\leq \varepsilon (KT \exp(2K(T-t)) + T) \\ &\quad + \mathbb{E} \left[\sup_{t \leq s \leq T} 2\sqrt{\varepsilon} \int_t^s (X^{\varepsilon,t,x}(r) - \chi^{t,x}(r)) dW(r) \right]. \end{aligned}$$

It follows from the B-D-G inequality that there exists a constant K_1 such that

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)|^2 \right] &\leq \varepsilon (KT \exp(2K(T-t)) + T) \\ &\quad + 2K_1 \sqrt{\varepsilon} \left(\alpha + \frac{1}{\alpha} \mathbb{E} \left[\sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)|^2 \right] \right), \end{aligned}$$

where α is a positive constant to be determined.

Consequently, choosing some α we obtain that

$$\mathbb{E} \left[\sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)|^2 \right] \leq \varepsilon K_2,$$

where K_2 only depends on T, K, K_1 and is independent of ε, t, x . \square

Remark 10. As a consequence of Lemma 7, the solution of the $X^{\varepsilon,t,x}$ converges to the deterministic path $\chi^{t,x}$ in L^2 .

Now consider the following deterministic equations

$$\begin{cases} \chi^{t,x}(s) = x + \int_t^s b(r, \chi^{t,x}(r)) dr. \\ Y^{t,x}(s) = g(X^{t,x}(T)) + \int_s^T f(r, X^{t,x}(r), Y^{t,x}(r), 0) dr \\ \quad + K^{t,x}(T) - K^{t,x}(r) \\ Y^{t,x}(s) \geq h(s, X^{t,x}(s)), \quad t \leq s \leq T \\ \int_t^T (Y^{t,x}(s) - h(s, X^{t,x}(s))) dK^{t,x}(s), \end{cases}$$

(3.5)

We have the following Lemma

Lemma 11. *For all $\varepsilon \in (0, 1]$, there exists a constant $C > 0$, independent of x , t and ε , such that*

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \leq s \leq T} |Y^{t,x}(s) - Y^{\varepsilon,t,x}(s)|^2 \right] + \mathbb{E} \left[\int_t^T |Z^{\varepsilon,t,x}(s)|^2 ds \right] \\ & \leq C \mathbb{E} \left[\left(\sup_{t \leq s \leq T} |\widehat{\Xi}^{\varepsilon,t,x}(s)|^2 \right)^{\frac{1}{2}} + |\widehat{\Xi}^{\varepsilon,t,x}(T)|^2 + \int_t^T |\widehat{\Xi}^{\varepsilon,t,x}(s)|^2 ds \right] \end{aligned} \quad (3.6)$$

where $\widehat{\Xi}^{\varepsilon,t,x}(s) = X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)$, $0 \leq t \leq s \leq T$, P -a.s.

Proof. It follows from Itô's formula that

$$\begin{aligned} & \mathbb{E} \left[|Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)|^2 \right] + \mathbb{E} \left[\int_t^T |Z^{\varepsilon,t,x}(s)|^2 ds \right] \\ & = \mathbb{E} [g(X^{\varepsilon,t,x}(T)) - g(\chi^{t,x}(T))] + 2\mathbb{E} \left[\int_t^T \widehat{f}(s) (Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)) ds \right] \\ & \quad + 2\mathbb{E} \left[\int_t^T (Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)) d(K^{\varepsilon,t,x}(s) - K^{t,x}(s)) \right], \end{aligned}$$

where

$$\widehat{f}(s) = f(s, X^{\varepsilon,t,x}(s), Y^{\varepsilon,t,x}(s), Z^{\varepsilon,t,x}(s)) - f(s, \chi^{t,x}(s), Y^{t,x}(s), 0).$$

Hence

$$\begin{aligned} & \mathbb{E} \left[|Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)|^2 \right] + \mathbb{E} \left[\int_t^T |Z^{\varepsilon,t,x}(s)|^2 ds \right] \\ & \leq K \mathbb{E} [|X^{\varepsilon,t,x}(T) - \chi^{t,x}(T)|^2] \\ & \quad + 2K \mathbb{E} \left[\int_t^T [|X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)| + |Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)| + |Z^{\varepsilon,t,x}(s)|] (Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)) ds \right] \\ & \quad + 2\mathbb{E} \left[\int_t^T (Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)) d(K^{\varepsilon,t,x}(s) - K^{t,x}(s)) \right]. \end{aligned} \quad (3.7)$$

Since that $K^{\varepsilon,t,x}$ (respectively $K^{t,x}$) grows when $Y^{\varepsilon,t,x}(s) = h(s, X^{\varepsilon,t,x}(s))$ (respectively $Y^{t,x}(s) = h(s, X^{t,x}(s))$) only, and $Y^{\varepsilon,t,x}(s) \geq h(s, X^{\varepsilon,t,x}(s))$ and $Y^{t,x}(s) \geq h(s, \chi^{t,x}(s))$ for

$s \in [t, T]$, P -a.s.

$$\begin{aligned}
& \int_t^T (Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)) \, d(K^{\varepsilon,t,x}(s) - K^{t,x}(s)) \\
&= \int_t^T (h(s, X^{\varepsilon,t,x}(s)) - Y^{t,x}(s)) \, dK^{\varepsilon,t,x} + \int_t^T (h(s, \chi^{t,x}(s)) - Y^{\varepsilon,t,x}(s)) \, dK^{t,x}(s) \\
&\leq \int_t^T |h(s, X^{\varepsilon,t,x}(s)) - h(s, \chi^{t,x}(s))| \, d(K^{\varepsilon,t,x}(s) + K^{t,x}(s)) \\
&\leq K \int_t^T |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)| \, d(K^{\varepsilon,t,x}(s) + K^{t,x}(s)) \\
&\leq K \sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)| \int_t^T d(K^{\varepsilon,t,x}(s) + K^{t,x}(s))
\end{aligned} \tag{3.8}$$

Taking the expectation on two sides of (3.8), we have

$$\begin{aligned}
& \mathbb{E} \left[\int_t^T (Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)) \, d(K^{\varepsilon,t,x}(s) - K^{t,x}(s)) \right] \\
&\leq K \left(\mathbb{E} \left[\sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)|^2 \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\int_t^T d(K^{\varepsilon,t,x}(s) + K^{t,x}(s))^2 \right] \right)^{\frac{1}{2}} \\
&\leq K \left(\mathbb{E} \left[\sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - \chi^{t,x}(s)|^2 \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[(K^{\varepsilon,t,x})^2(T) + (K^{t,x})^2(T) \right] \right)^{\frac{1}{2}}
\end{aligned}$$

It follows from the assumption on $b, \varepsilon \in (0, 1]$ and Lemma 1 that $\mathbb{E} \left[(K^{\varepsilon,t,x}(T))^2 + (K^{t,x}(T))^2 \right]$ is bounded denoted by a positive constant M independent of ε . Now we turn back to (3.7). Thanks to the inequality $ab \leq \frac{a^2+b^2}{2}$ and Grownwall's lemma, we deduce immediately that

$$\begin{aligned}
& \mathbb{E} \left[|Y^{\varepsilon,t,x}(s) - Y^{t,x}(s)|^2 \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T |Z^{\varepsilon,t,x}(s)|^2 \, ds \right] \\
&\leq K' \mathbb{E} \left[\left(\sup_{t \leq s \leq T} |\widehat{\Xi}^{\varepsilon,t,x}(s)|^2 \right)^{\frac{1}{2}} + |\widehat{\Xi}^{\varepsilon,t,x}(T)|^2 + \int_t^T |\widehat{\Xi}^{\varepsilon,t,x}(s)|^2 \, ds \right],
\end{aligned}$$

where K' depends on K, M and T . The proof is complete. \square

Remark 12. As a consequence of Lemmas 9 and 11, we get

$$\|G^\varepsilon(\varphi) - G(\varphi)\| \leq \sqrt{\varepsilon} K'', \quad \varphi \in C([0, T] : \mathbf{R}^n)$$

where K'' is a constant related to K' .

Now we are able to give the proof of Theorem 8:

Proof. By virtue of the contraction principle Theorem 4.2.23, page 133 in [2], we just need to show that G^ε , $\varepsilon \in (0, 1]$ are continuous and $\{G^\varepsilon\}$ converges uniformly to G on every compact of $C([0, T] : \mathbf{R}^n)$, as ε tends to zero. As a matter of fact, since u^ε is continuous by Proposition 3, it is not hard to prove that G^ε is also continuous. Next let us show the uniform convergence of the mapping G^ε . It follows from Remark 12. \square

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